Simulation Study on Methods for Modeling Elastic Collisions

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Introduction

This paper is focused on investigating and describing collision processes and comparing the numerical results of different ODE solvers. The collisions are modeled on a system of two points of mass moving along circular paths. The equations of motion are derived using the Euler-Lagrange method. All the simulations are done in MATLAB, and the derivations of the equations are done in Maple. Explicit Euler and Heun methods are implemented, and the results of both solvers were compared to the ode45 solver of MATLAB, which is used as reference due to complexity of the equations of movement of this coupled system.

The main question that this work aims to answer is how the different algorithms for solving differential equations behave when altered in a way that the collisions can be modeled.

The idea of this work is based on the problem of detecting neighborhoods of two objects while a movement occurs. As seen in the Figure 1, we observe mass points m_1 and m_2 moving along a certain path. Both of these points have fixed circular neighbourhoods U_1 and U_2, and cannot find each other in their respective neighbourhoods. When the distance between these two points is smaller than the radius of any of the circles, a certain type of collision happens. We will simplify this generalized idea in the next chapter. The distance between two points needs to be calculated at all times in this model, and has to be stopped when two objects find themselves within a given radius of each other. This was implemented in two different differential equation solvers, explicit Euler and Heun.

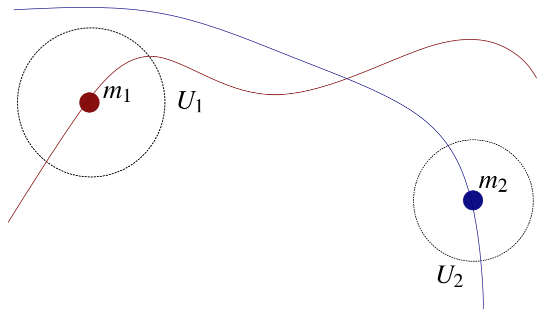


Figure 1: A sketch of the generalised idea of this work. The points (blue and red) are moving along their paths, and have masses m_1 and m_2, with the respective neighbourhoods $U_{1,2}$.

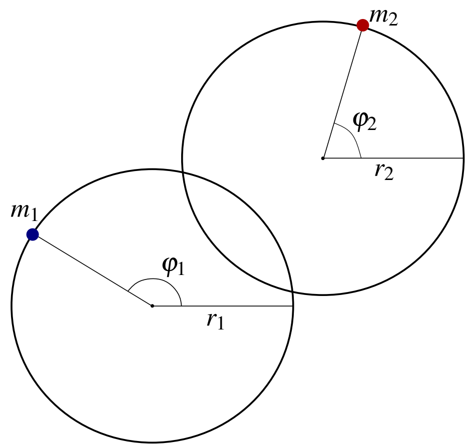


Figure 2: A sketch of the system setup, the blue dot represents the point mass m_1 and the red dot the point mass m_2. Movement occurs in circular paths, with position vectors φ_1 and φ_2, both with fixed radus $r_{1,2}$.

1 Description of the System

The system observed is a system of two mass points m_1 and m_2 moving along a circular path in a two-dimensional space. This can be seen in the Figure 2. Each of them has a fixed radius of movement and can bump into each other at times.

The equations of motion are derived using the Euler-Lagrange method. The kinetic energy T_i of each point mass is

$$T_i = \frac{1}{2} m_i \dot{r}_i(t), \quad i = 1,2. \tag{1}$$

The variable $r_i(t)$ represents the position vector of the respective points of mass. Since we are observing a circular motion, the position can be restricted to circular motion, with the displacement of the two origins of the circles by a vector $(d_x, d_y)^T$, so the constraints do not have to be additionally considered,

$$r_1(t) = \begin{pmatrix} r_1 \cos\varphi_1 \\ r_1 \sin\varphi_1 \end{pmatrix} \quad \text{and} \quad r_2(t) = \begin{pmatrix} r_2 \cos\varphi_2 + d_x \\ r_2 \sin\varphi_2 + d_y \end{pmatrix}.$$

The points are attracted to each other by gravity force so the potential energy is observed as

$$U = -\frac{G m_1 m_2}{d}, \tag{2}$$

with $d(t) = \|r_1(t) - r_2(t)\|$ representing the euclidean distance between the two points, and $G = 6{,}7 \cdot 10^{-11} \mathrm{m}^3 \, \mathrm{kg}^{-1} \, \mathrm{s}^{-2}$. The equations of motion are obtained by using the Euler-Lagrange Equation

$$\frac{\partial L}{\partial \varphi_i} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\varphi}_i}\right) = 0. \tag{3}$$

The initial conditions are set by altering the position angles φ_1 and φ_2, as well as the angular velocities $\dot{\varphi}_1$ and $\dot{\varphi}_2$, which are both time-dependent.

2 Collisions

The model calculates the distance between the two points at all times and when they find each other in the defined neighborhood of each other, a collision is modeled. The reason why the distance for impact to happen was not set to zero is that the potential energy used in this model is the gravitational energy, see (2), where we divide with the distance. In order to avoid division with zero, we set the distance for impact to occur to be $d_{min} = 0.2$m. The collisions are modeled as elastic collisions, and the initial conditions are updated after each impact based on the Law of conservation of momentum and energy, as can be seen in (4).

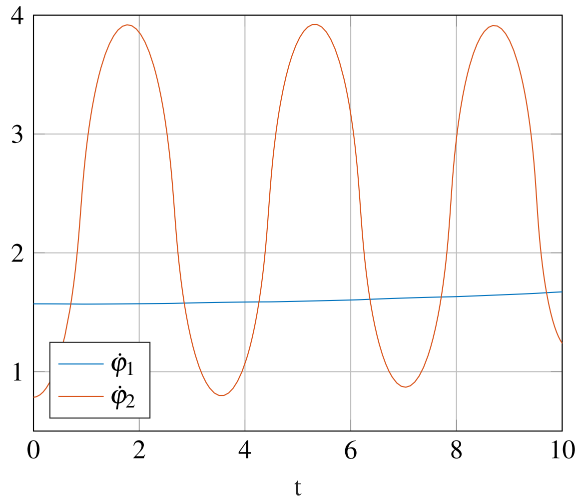
$$m_1 \dot{\varphi}_1(t^-) + m_2 \dot{\varphi}_2(t^-) = m_1 \dot{\varphi}_1(t^+) + m_2 \dot{\varphi}_2(t^+) \tag{4}$$
$$m_1 \dot{\varphi}_1(t^-)^2 + m_2 \dot{\varphi}_2(t^-)^2 = m_1 \dot{\varphi}_1(t^+)^2 + m_2 \dot{\varphi}_2(t^+)^2$$

3 Simulation Studies

The step size of the solving methods is varied and the quality of the solutions is compared to the results of ode45 from MATLAB. Without initial velocity, the simulations take some time befoe a collision occurs. When an initial velocity is introduced properly, the colisions happen more often, and this is the case we are also going to observe.

The parameters are $m_1 = 10^6$ kg, $m_2 = 10^3$ kg, the radii of both paths are $r_1 = r_2 = 1$ m. Since we are dealing with the gravitational energy as a potential energy, the two points need to have a large mass in order for any movement to occur, but using large numbers is prone to numeric errors, so the model was built in 10^6 scale.

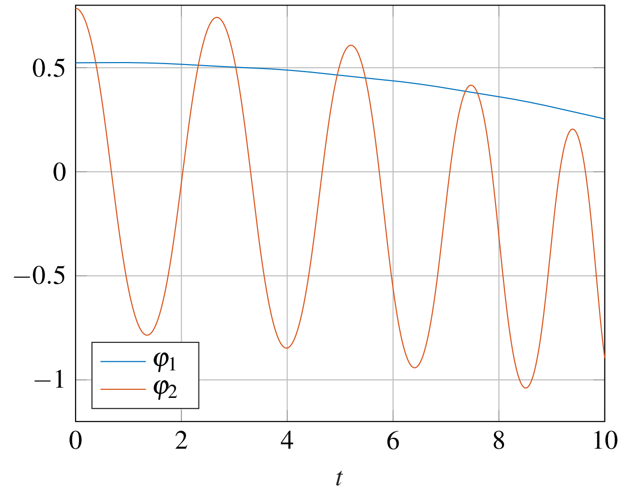
Before decribing the results from the explicit solvers, we will present the results obtained with MATLAB solver ode45. Here we can see that we are expecting a sinusoidal movement for the second point of mass, m_2, which is moving around the much heavier point m_1, which has a very restriced angular movement.



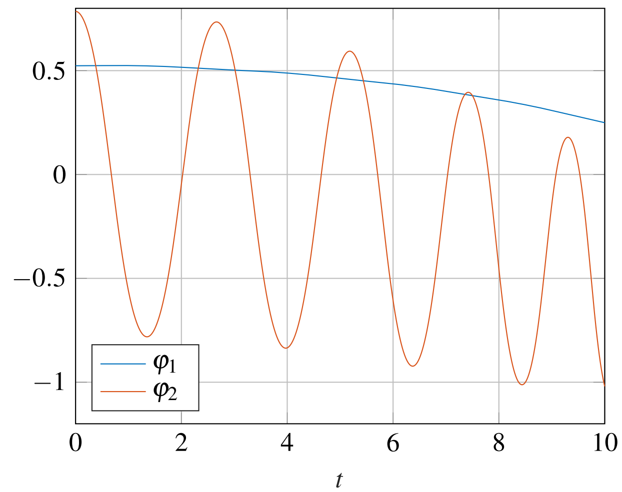
(a)

Figure 3: Simulation with MATLAB ode45, time plot of angular position, simulation time 10s.

First, we observed the movement of the particles with no initial velocity given. The initial conditions are set to $\varphi_{1,0} = \frac{\pi}{6}$ and $\varphi_{2,0} = \frac{\pi}{4}$, $\dot{\varphi}_{1,0} = 0$ and $\dot{\varphi}_{2,0} = 0$. The large difference between the two masses is necessary, otherwise the movement does not happen with no initial velocity. In these simulations, no collisons happened which was useful for heuristic comparison of the methods. The explicit Euler method showed some stable movement at step size $h = 0.001$ and the Heun method at step size $h = 0.01$. These results can be observed in the Figure 4(a) for Euler and Figure 4(b) for Heun method.



(a)



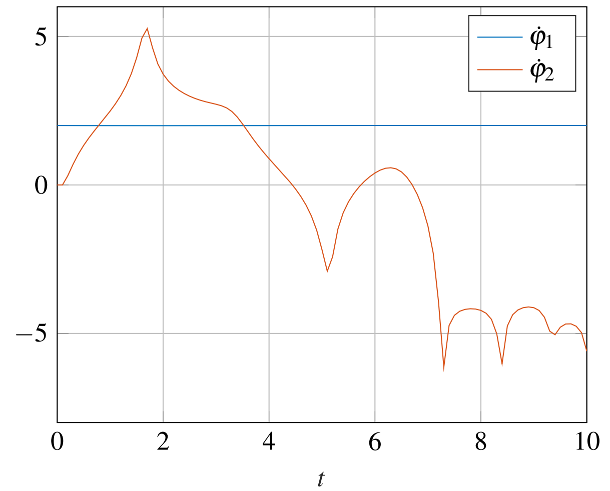
(b)

Figure 4: (a) Simulation with explicit Euler method, with step size $h = 0.001$, (b) Simulation with Heun method, with step size $h = 0.01$. Time plot of angular position. Both with simulation time 10s.

This result was used to determine the methods that will be compared when the collisions are introduced to the system. As we can see, when using the Heun method, which is also an explicit method of solving, we can observe a similar solution to the Euler method, but with less steps and lower computational time.

After this, we introduce the initial velocity which improves the chances of a collision happening. The initial conditions are set to $\varphi_{1,0} = \varphi_{2,0} = \frac{\pi}{4}$, $\dot{\varphi}_{1,0} = 2\mathrm{s}^{-1}$ and $\dot{\varphi}_{2,0} = 0$. Other parameters remain the same as in the previous section. We are modeling the elastic collision, and therefore expect the point with a lower mass, in this case m_2 to move in the opposite direction after the impact, which can usually be observed in a form of a jump on the time-velocity plot.

If the step size is large, when using the explicit Euler method, no collisions are detected, with the initial conditions mentioned below, as can be seen in figure 5.



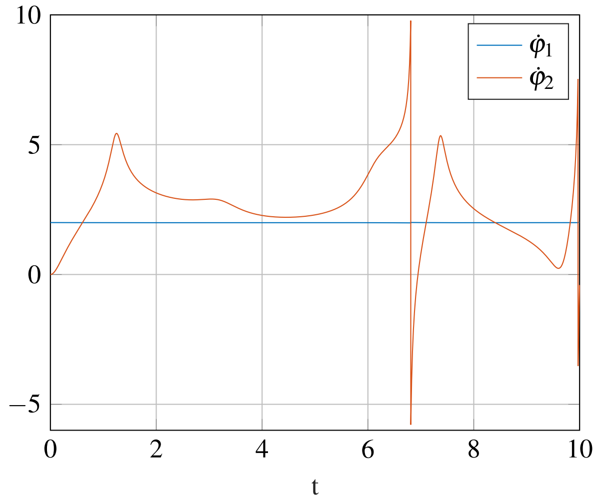
(a)

Figure 5: Simulation with explicit Euler method, with step size $h = 0.1$, for the simulation time of 10s. Time-plot of angular velocity.

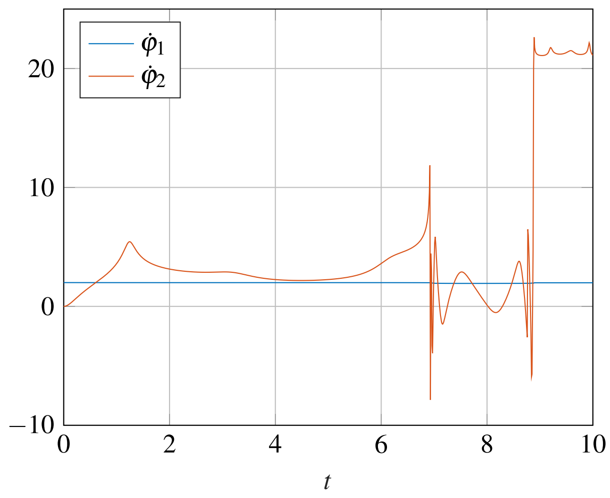
In the Figure 6 we can observe the motion with a finer step size. A collision is detected in this case, at $t = 6.809\,\mathrm{s}$ and $t = 9.970\,\mathrm{s}$. Since the step size is finer, the solver is more sensitive to the neighborhood radius $d_{min} = 0,2\mathrm{m}$, and can detect the collisions more precisely.

When using Heun method with step size $h = 0.01$, collisions are detected at $t = 6.920\,\mathrm{s}$ and at $t = 8.760\,\mathrm{s}$.

The difference between the two methods increases after each collision, which is obvious for the mass point with lower mass, since it is exposed to more movement. The mass point with larger mass does not change its path a lot after collisions. This error can be repaired by decreasing the step size.



(a)



(b)

Figure 6: (a) Simulation with explicit Euler method, with step size $h = 0.001$. Time-plot of angular velocity. (b) Simulation with Heun method, with step size $h = 0.01$. Time-plot of angular velocity. Both for the simulation time of 10s.

4 Conclusion

This work was focused on comparing two explicit methods for solving ordinary differential equations, both modified to calculate the distance between two moving objects and stop the calculation if they find themselves within a given radius of each other. The system observed in this case were two points of mass in a two dimensional space, attracted to each other by gravity. The equations of motion resulted in a system of two nonlinear ordinary differential equations of second order. They were numerically solved by using explicit Euler and Heun methods.

In order to observe solutions, we simulated the motion without collisions by using ode45 from MATLAB. This served as a reference simulation for the rest of the work. In the next step of the computation, we simulated the motion with explicit Euler and Heun, where we varied the step size in order to determine the best simulations. As expected, the simulations both exibited better results with a finer step size. Stable solutions were obtained for explicit Euler method for step size $h = 0.001$ and the Heun method gave similar results at step size $h = 0.01$. Instabilities and unusual behavior occurred after some of the collisions happened, which can be observed on the figures.

The equations of motion for this system are quite complicated and an analytic solution, if it exists, is out of scopes of this work. We only used numerical data for comparison. The lack of an analytic solution made if difficult to compare the results, since there was no reference, and we had to settle with choosing a good solution and doing a comparison. This could be improved by altering the system parameters so that the equations of motion have a lower complexity.

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